Further Pure 1 - June 2013

(1) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = 10$

$f(x) = x^3 - x^2 + 4x - 900$

$f(10) = 10^3 - 10^2 + 4(10) - 900 = 40$

$f'(x) = 3x^2 - 2x + 4$

$f'(10) = 3(10)^2 - 2(10) + 4 = 284$

$\rightarrow x_{n+1} = 10 - \dfrac{40}{284} = 9.85915\ldots$

$= 9.859$ (4 sf)

(2) a) i) $\begin{bmatrix} p & 2 \\ 4 & p \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} p-3 & 1 \\ 2 & p-3 \end{bmatrix}$

ii) $\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$

$\begin{bmatrix} p & 2 \\ 4 & p \end{bmatrix} \begin{bmatrix} 3p+4 & p+6 \\ 12+2p & 4+3p \end{bmatrix} \rightarrow \begin{bmatrix} 3p+4 & p+6 \\ 12+2p & 4+3p \end{bmatrix}$

b) $A - B + AB = \begin{bmatrix} p-3 & 1 \\ 2 & p-3 \end{bmatrix} + \begin{bmatrix} 3p+4 & p+6 \\ 12+2p & 4+3p \end{bmatrix}$

$= \begin{bmatrix} 4p+1 & p+7 \\ 14+2p & 1+4p \end{bmatrix}$ Needs to be $k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\rightarrow p + 7 = 0 \rightarrow p = -7$

$\rightarrow \begin{bmatrix} -27 & 0 \\ 0 & -27 \end{bmatrix} = -27 I$ $\quad k = -27$

(3) a) cos: $\theta = 360n \pm a$

Key angle $= \cos^{-1}(\cos 65) = 65 = a$

$\rightarrow \quad 5x + 40 = 360n \pm 65$

$\rightarrow \quad 5x = 360n - 40 \pm 65$

$\rightarrow \quad x = 72n - 8 \pm 13$

$\rightarrow \quad x = 72n + 5 \quad , \quad x = 72n - 21$

b) $\cos(\pi/4) = 1/\sqrt{2}$

$\sin(\pi/12) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$

$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3} - 1}{2}\right)$

$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

$\uparrow \qquad \uparrow \qquad \uparrow$

$\cos(\pi/4) \qquad \cos(\pi/6) \qquad \cos(2\pi/3)$

$\rightarrow \quad \cos(\pi/4)\left(\cos(\pi/6) + \cos(2\pi/3)\right)$

$a = 1/6, \quad b = 2/3$

(4) a) i) $(z - 2i)^* = (x + yi - 2i)^*$

$= (x - yi + 2i)$

ii) $x - yi + 2i = 4iz + 3$

$\rightarrow \quad x - yi + 2i = 4i(x + yi) + 3$

$\rightarrow \quad x - yi + 2i = 4xi - 4y + 3$

$\boxed{\text{REAL}} \quad x = -4y + 3$

$\boxed{\text{Imag}} \quad -y + 2 = 4x$

Real $\times 4 \rightarrow \quad 4x = -16y + 12$

Equate: $\quad -16y + 12 = -y + 2$

$\rightarrow \quad 10 = 15y \quad \rightarrow \quad y = 10/15 = 2/3$

$x = -4(2/3) + 3 = 1/3$

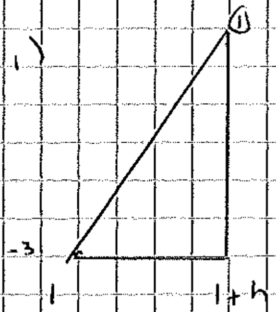
$\Rightarrow z = 1/3 + 2/3\, i$

b) Sum of roots $= -10i$

$(p + qi) + (p - qi) = 2p$

$\therefore p$ cannot be real if $p - qi$ is a root.

(5) a) i)



① $y = 2(1+h)^2 - 5(1+h)$

$\Rightarrow y = 2(1 + 2h + h^2) - 5 - 5h$

$\Rightarrow y = 2 + 4h + 2h^2 - 5 - 5h$

$\Rightarrow y = 2h^2 - h - 3$

$$\text{Gradient} = \frac{2h^2 - h - 3 - (-3)}{h}$$

$$= 2h - 1$$

ii) As $h \to 0$, gradient of tangent $\to 2(0) - 1 = -1$

Gradient of $x + y = 0$ $= -1$

$\therefore$ Lines are parallel

b) $$I = \int_1^m 2x^{-2} - 5x^{-3} = \left[-2x^{-1} + \tfrac{5}{2}x^{-2}\right]_1^m$$

$$= \left[\frac{-2}{x} + \frac{5}{2x^2}\right]_1^m = \left(\frac{-2}{m} + \frac{5}{2m^2}\right) - \left(\frac{-2}{1} + \frac{5}{2}\right)$$

$$= \frac{-2}{m} + \frac{5}{2m^2} - 0.5$$

As $m \to \infty$, $-2/m$ & $5/2m^2 \to 0$

$\therefore \int \to -0.5$

(6) a) $\alpha + \beta = -3/2 \qquad \alpha\beta = -6/2 = -3$

b) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (-3/2)^3 - 3(-3)(-3/2)$

$= -27/8 - 27/2 = -135/8$

$\boxed{\text{Sum}}$ $\alpha + \frac{\alpha}{\beta^2} + \beta + \frac{\beta}{\alpha^2}$

$= \alpha + \beta + \frac{\alpha^3}{\alpha^2\beta^2} + \frac{\beta^3}{\alpha^2\beta^2}$

$= (\alpha + \beta) + \frac{\alpha^3 + \beta^3}{(\alpha\beta)^2}$

$= -3/2 + \frac{-135/8}{(-3)^2} = -27/8$

$\boxed{\text{PRODUCT}}$ $\left(\alpha + \frac{\alpha}{\beta^2}\right)\left(\beta + \frac{\beta}{\alpha^2}\right) = \alpha\beta + \frac{\alpha\beta}{\alpha^2} + \frac{\alpha\beta}{\beta^2} + \frac{\alpha\beta}{\alpha^2\beta^2}$

$= \alpha\beta + \frac{\beta}{\alpha} + \frac{\alpha}{\beta} + \frac{1}{\alpha\beta}$

$= \alpha\beta + \frac{\alpha^2 + \beta^2}{\alpha\beta} + \frac{1}{\alpha\beta}$

$= \alpha\beta + \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} + \frac{1}{\alpha\beta}$

$= -3 + \frac{(-3/2)^2 - 2(-3)}{-3} + \frac{1}{-3}$

$= -73/12$

$x^2 - \boxed{\text{Sum}}\,x + \boxed{\text{PRODUCT}} = 0$

$\Rightarrow x^2 + 27/8\,x - 73/12 = 0$

$(\times 24) \Rightarrow 24x^2 + 81x - 146 = 0$

(7) a) Let $f(x) = 4x^3 - x - 540{,}000$

$f(51) = 4(51)^3 - (51) - 540{,}000 = -9447$

$f(52) = 4(52)^3 - (51) - 540{,}000 = 22{,}380$

Sign change, $\therefore$ root lies between 51 & 52

b) i) $S_n = \sum 4r^2 - 4r + 1$

$= 4\sum r^2 - 4\sum r + \sum 1$

$= \frac{2}{3} n(n+1)(2n+1) - 2n(n+1) + n$

$= \frac{n}{3}\left[2(n+1)(2n+1) - 6(n+1) + 3 \right]$

$= \frac{n}{3}\left[2\left[2n^2 + n + 2n + 1\right] - 6n - 6 + 3 \right]$

$= \frac{n}{3}\left[4n^2 + 6n + 2 - 6n - 3 \right]$

$= \frac{n}{3}\left[4n^2 - 1 \right] \quad \Rightarrow k = 4$

ii) $6S_n = 2n\left[4n^2 - 1\right]$

$= 2n\,(2n+1)(2n-1)$

$= (2n-1)\,2n\,(2n+1)$

$=$ product of 3 consecutive integers.

c) Let odd numbers be $2n-1$

eg $n = \{1 \quad 2 \quad 3 \quad 4 \quad \ldots$

numbers $\} 1 \quad 3 \quad 5 \quad 7 \quad \ldots$

$\therefore \sum_{1}^{n} (2n-1)^2 > 180{,}000$

$\Rightarrow \frac{n}{3}(4n^2 - 1) > 180{,}000$

Try values: $n = 51 \rightarrow 176{,}851$ (too small)

$n = 52 \rightarrow 187{,}460$ (too big)

$\therefore n = 52$ is smallest value of n.

(8) a) Stretch, SF 3 in y-direction

$\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

b) i) $y = \sqrt{3}x \Rightarrow y = (\tan 60)\,x$

$\rightarrow \begin{bmatrix} \cos(120) & \sin(120) \\ \sin(120) & -\cos(120) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

ii) Need Matrix (2) × Matrix (1)

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1/2 & 3\sqrt{3}/2 \\ \sqrt{3}/2 & 3/2 \end{bmatrix}$$

(9) a) Denominator : $(x-3)(x+1)$

$\rightarrow x = 3$ & $x = -1$ are asymptotes

As $x \rightarrow \infty$, $y \rightarrow 1/1 \rightarrow y = 1$ is asymptote.

b) i) $y = k \rightarrow k = \dfrac{x^2 - 2x + 1}{x^2 - 2x - 3}$

$\rightarrow k = \dfrac{(x-1)(x-1)}{(x-3)(x+1)}$ NOT NEEDED!

$\rightarrow k(x^2 - 2x - 3) = x^2 - 2x + 1$

$\rightarrow kx^2 - 2kx - 3k = x^2 - 2x + 1$

$\rightarrow kx^2 - x^2 - 2kx + 2x - 3k - 1 = 0$

$\rightarrow (k-1)x^2 - 2(k-1)x - (3k+1) = 0$

ii) Real roots $\rightarrow b^2 - 4ac \geq 0$

$\rightarrow [-2(k-1)]^2 + 4(k-1)(3k+1) \geq 0$

$\rightarrow 4[k^2 - 2k + 1] + 4[3k^2 + k - 3k - 1] \geq 0$

(÷4) $\rightarrow k^2 - 2k + 1 + 3k^2 \cancel{} - 2k - 1 \geq 0$

$\rightarrow 4k^2 - 4k \geq 0$

(÷4) $\rightarrow k^2 - k \geq 0$

iii) At stationary points, $k^2 - k = 0$

$\rightarrow k(k-1) = 0$

$k = 1$

$k = 0$

$\rightarrow y = 1$ = Asymptote!

$\rightarrow y = 0$

So $y = 1$ NOT stationary point

When $y = 0$, $x \to 0 = \dfrac{x^2 - 2x + 1}{x^2 - 2x - 3}$

$$\to 0 = (x-1)(x-1)$$

$$\to x = 1$$

$\therefore$ Only 1 stationary point at $(1, 0)$

c)